

# ECON 7010 - MACROECONOMICS I

Fall 2015

Notes for Lecture #4

Today- Cake eating problem:

- Stochastic cake eating
- Discrete choice dynamic programming
- Necessary conditions for DPP solution

## Stochastic Cake Eating Problem

- $V(w, \varepsilon) = \max_{w'} \varepsilon u(w - w') + \beta E_{\varepsilon'|\varepsilon} V(w', \varepsilon'), \forall (w, \varepsilon)$
- State variables:  $w$ =size of cake;  $\varepsilon$ =taste shock,  $\varepsilon \in \{\varepsilon_H, \varepsilon_L\}$
- Control variables =  $w'$  (or  $c$ )
- Transition equations:  $w' = w - c, \underbrace{\Pi}_{2 \times 2}$ : transition matrix
  - $\pi_{ij} = Pr(\varepsilon' = \varepsilon_j | \varepsilon = \varepsilon_i), i = L, H, j = L, H$
  - $\sum_j \pi_{ij} = 1$  for  $i = L, H$  (sum over  $j$ )
  - This is a first order Markov process - only one period in the past matters for conditional prob today
- Policy function:  $w' = \varphi(w, \varepsilon)$ 
  - consumption a function of how much cake and the taste shock  $\rightarrow$  eat more when  $\varepsilon_H$ , less when  $\varepsilon_L$
- in terms of empirical research, the policy function is what you care about  $\rightarrow$  it's the stepping stone between the transition equation and the value function
- $\pi_{LL}$  near 1  $\Rightarrow$  eat cake now before get low and stuck there
- $\pi_{LL}$  near 0  $\Rightarrow \pi_{LH}$  near 1  $\Rightarrow$  wait and eat cake when  $\varepsilon_H$
- decision is based on the present  $\varepsilon$  and also what that  $\varepsilon$  means for the probability of different future  $\varepsilon$ 's and expected future utility
- FOC:
  - $\frac{\partial V}{\partial w'} : \underbrace{\varepsilon}_{\text{For a particular value of } \varepsilon} u'(w - w') = \beta E_{\varepsilon'|\varepsilon} \underbrace{V_{w'}(w', \varepsilon')}_{\text{Random variable b/c } \varepsilon' \text{ random}}, \forall (w, \varepsilon)$
  - e.g.,  $\varepsilon = \varepsilon_H$ :
  - $\varepsilon_H u'(w - w') = \beta \{ \pi_{HL} \underbrace{V_{w'}(w', \varepsilon_L)}_{\text{value of low next period}} + \pi_{HH} \underbrace{V_{w'}(w', \varepsilon_H)}_{\text{value of high next period}} \}$
  - Agent knows  $\varepsilon$  when choose  $w'$ , so  $w'$  fixed, but  $w'$  a function of  $\varepsilon$ ;  $w' = \varphi(w, \varepsilon)$
  - FOC for low state:  $\varepsilon_L u'(w - w') = \beta \{ \pi_{LL} V_{w'}(w', \varepsilon_L) + \pi_{LH} V_{w'}(w', \varepsilon_H) \}$
  - Note:  $w'$  here different than  $w'$  above b/c chose to consume more at  $\varepsilon_H$  than  $\varepsilon_L$
- Euler equation

– Using the envelope theorem, we know:

\*  $V_w(w, \varepsilon) = \varepsilon u'(c)$  (students can work this out on their own)

\*  $\implies V_{w'}(w', \varepsilon') = \varepsilon' u'(c')$

–  $\implies$  Euler is:  $\varepsilon u'(c) = \underbrace{\beta E_{\varepsilon'|\varepsilon} \varepsilon' u'(c')}_{\text{Discounted expected MU}}$

– can write:  $c = w - w', c' = w' - w''$  (where " is two periods ahead)

– Substituting in the policy function ( $w' = \varphi(w, \varepsilon)$ ) and  $c = w - w'$  get:

–  $\varepsilon u'(w - \varphi(w, \varepsilon)) = \beta E_{\varepsilon'|\varepsilon} \varepsilon' u'[\varphi(w, \varepsilon) - \varphi(\varphi(w, \varepsilon), \varepsilon')]$

–  $c = w' - w = w - \varphi(w, \varepsilon) \equiv \phi(w, \varepsilon) \rightarrow$  consumption depends on  $\varepsilon \implies c' = \phi(w', \varepsilon')$

• Comparative statics with the stochastic dynamic programming problem:

– Let  $G(w', \varepsilon_H) = \varepsilon_H u'(w - w') - \beta [\pi_{HL} \varepsilon_L u'(w' - w'') + \pi_{HH} \varepsilon_H u'(w' - w'')] = 0$

– Applying the IFT:  $\frac{\partial w'}{\partial \varepsilon_H} = \frac{-G_2}{G_1} = -\frac{u'(w-w') - \beta \pi_{HH} u'(w'-w'')}{-\varepsilon_H u''(w-w') - \beta [\pi_{HL} \varepsilon_L u''(w'-w'') + \pi_{HH} \varepsilon_H u''(w'-w'')]}$

– Now try to sign the derivative...

\* Intuitively, we think that the agent should save less if he is in the high state today and the value of the high state increases. This means the marginal utility of consumption increases today and so to keep equality across periods we need increase consumption this period and lower consumption next period as that will lower the MU of cons today and increase the MU cons tomorrow.

\* By the FOC, we know that:  $\varepsilon_H u'(w - w') = \beta [\pi_{HL} \varepsilon_L u'(w' - w'') + \pi_{HH} \varepsilon_H u'(w' - w'')]$

\*  $\implies \varepsilon_H u'(w - w') - \beta \pi_{HH} \varepsilon_H u'(w' - w'') = \pi_{HL} \varepsilon_L u'(w' - w'')$

\*  $\implies u'(w - w') - \beta \pi_{HH} u'(w' - w'') = \underbrace{\frac{\pi_{HL} \varepsilon_L u'(w' - w'')}{\varepsilon_H}}_{>0}$

\* Thus we know that the numerator in the partial derivative is positive.

\* And since  $u'' < 0$ , we know the denominator is negative.

\* Thus we know that  $\frac{\partial w'}{\partial \varepsilon_H} > 0$ . That is,  $w'$  falls as  $\varepsilon_H$  increases

• But  $\varepsilon$  doesn't always have to be in the policy function

• Whether or not it is, depends upon how expectations are made

• Two alternative are:

• 1) Make it additive:  $V(w, \varepsilon) = \max_{w'} \varepsilon + u(w - w') + \beta E_{\varepsilon'|\varepsilon} V(w', \varepsilon')$

– make it additive rather than multiplicative so that  $\varepsilon$  has no effect at the margin, though it does affect overall utility

– no  $\varepsilon$  in Euler (on left)(when take  $\frac{\partial V}{\partial w'}$ ,  $\varepsilon$  left out)

• 2) Agent doesn't know shock when makes choice of control:

–  $V(w, \varepsilon_{-1}) = \max_{w'} E_{\varepsilon|\varepsilon_{-1}} \underbrace{\{\varepsilon u(w - w') + \beta V(w', \varepsilon)\}}_{\text{both depends on } E_{\varepsilon|\varepsilon_{-1}}}, \forall (w, \varepsilon_{-1})$

\* The FOC for this problem:  $E_{\varepsilon|\varepsilon_{-1}} [\varepsilon u'(w - w') - \beta V(w', \varepsilon)] = 0$

\* This implies  $E_{\varepsilon|\varepsilon_{-1}} \varepsilon u'(c) = \beta E_{\varepsilon|\varepsilon_{-1}} u'(c')$

\* Neither sides of the above equation are functions of  $\varepsilon$  (though they are functions of  $\varepsilon_{-1}$ )

\* the above is a model of making a decision today base on  $\varepsilon$  from a previous period

\* since utility today and continuation value both uncertain in same way, the decision will only depend on the expected value

\* always need to specify who knows what, when, why

Discrete Choice Cake Eating Problem: (an example of an optimal stopping problem)

- control: {eat cake, leave cake}  $\rightarrow$  binary (0,1 choice)
  - $z \in \{1, 0\}$
- state:  $w, \varepsilon \Rightarrow$  know  $w$  and  $\varepsilon$  at the time of the decision
- transition:  $w' = \rho w$  if  $z = 0$  (grow/shrink leftover cake),  $w' = 0$  if  $z = 1$  (cake eaten in period 1, no  $w'$ )
- value function:  $V(w, \varepsilon) = \max\{\underbrace{V^0(w, \varepsilon)}_{\text{leave cake}}, \underbrace{V^1(w, \varepsilon)}_{\text{eat cake}}\}, \forall (w, \varepsilon)$ 
  - $V^0(w, \varepsilon) = \beta E_{\varepsilon'|\varepsilon} V(\rho w, \varepsilon')$
  - $V^1(w, \varepsilon) = \varepsilon u(w)$
- policy function:  $z(w, \varepsilon) \in \{0, 1\}, \forall (w, \varepsilon)$
- Choice depends on:
  - State variables: ( $w$  &  $\varepsilon$ ) b/c in state vector
  - Parameters:
    - \*  $\rho$ , b/c as  $\rho \uparrow$ , gain to waiting
    - \*  $\beta$ ,  $\beta \downarrow$  cost to waiting
    - \*  $\Pi$ : the transition matrix
- NOTE: No Euler equation in discrete case - eat or don't eat - it's not continuous
- e.g.,  $\rho = 1, \varepsilon \in \{\varepsilon_L, \varepsilon_H\}$ 
  - $z(w, \varepsilon_H) = 1, \forall w$ : nothing to wait for!
  - $z(w, \varepsilon_L) = \{0, 1\} \rightarrow$  wait if:  $\beta$  near 1 or  $\pi_{LH}$  sufficiently high
    - \* NOTE:  $w$  unimportant b/c its in both  $V^0$  and  $V^1$  decisions
    - \* How high does  $\pi_{LH}$  have to be to wait?
    - \* wait if:  $\varepsilon_L u(w) \leq \beta \{E_{\varepsilon'|\varepsilon_L} V(w, \varepsilon')\} = \beta \{\pi_{LH} \varepsilon_H u(w) + \pi_{LL} V(w, \varepsilon_L)\}$
    - \* B/c always eat in high, and assuming never eat in low (this is the RHS of the equality), know that:  $E_{\varepsilon'|\varepsilon_L} V(w, \varepsilon') = \pi_{LH} \varepsilon_H u(w) + \pi_{LL} \beta E_{\varepsilon'|\varepsilon_L} V(w, \varepsilon')$
    - \* Solving for  $V(w, \varepsilon') \Rightarrow E_{\varepsilon'|\varepsilon_L} V(w, \varepsilon') = \frac{\pi_{LH} \varepsilon_H u(w)}{1 - \beta \pi_{LL}}$
    - \* Thus, wait if  $\varepsilon_L u(w) \leq \beta \{E_{\varepsilon'|\varepsilon_L} V(w, \varepsilon')\} = \frac{\beta \pi_{LH} \varepsilon_H u(w)}{1 - \beta \pi_{LL}}$
    - \* Note that we can divide both sides by  $u(w)$ :  $\varepsilon_L \leq \frac{\beta \pi_{LH} \varepsilon_H}{1 - \beta \pi_{LL}}$
    - \* So, without growth in the size of the cake overtime, the decision rule is not a function of the size of the cake or the parameterization of the utility function.
    - \* NOTE: this is not the case if the size of the cake is growing - as on one of your HW problems.

General Dynamic Programming Problem:

- consider:  $V(s) = \max_{c \in C(s)} \underbrace{\sigma(c, s)}_{\text{the payoff of period one}} + \beta V(s'), \forall s \in S$

- state:  $s$
- control:  $c$
- transition equation:  $s' = \tau(s, c) \rightarrow$  what you have now ( $s$ ) and what you choose ( $c$ ) determines what you have tomorrow ( $s'$ )
- or you could write:
- (\*\*)  $V(s) = \max_{s' \in \Gamma(s)} \tilde{\sigma}(s, s') + \beta V(s'), \forall s \in S$
- policy function:  $s' = \phi(s)$
- The above is the general structure for the non-stochastic dynamic programming problem
- Existence of a solution:
  - Stokey and Lucas, *Recursive Methods in Economic Dynamics* (w/ Ed Prescott), [Theorem 4.6]
  - Adda-Cooper (page 24)
  - If:
    - \*  $\underbrace{\sigma(s, s')}_{\text{the payoff}}$  is continuous, real valued, and bounded (i.e., can be kept in a box)
    - \*  $0 < \beta < 1$  (i.e., there's discounting)
    - \*  $\Gamma(s)$  is non-empty, compact-valued ( $\Gamma(s)$  maps into itself), continuous
  - Then  $\exists$  a unique solution  $V(s)$  solving  $V(s) = \max_{s' \in \Gamma(s)} \tilde{\sigma}(s, s') + \beta V(s'), \forall s \in S$ 
    - \* Prove by applying a contraction mapping theorem
    - \* There exists a fixed point and the fixed point can be reached by iterating on an initial guess
    - \*  $\rightarrow$  unique sol'n found via Value Function Iteration
    - \*  $V_{i+1}(s) = \max_{s' \in \Gamma(s)} \tilde{\sigma}(s, s') + \beta V_i(s'), \forall s \in S$
    - \*  $\lim_{i \rightarrow \infty} V_i \rightarrow V^*$

Notes on cake.m:

- Go through program set up - parameters set, grid make, consumption and utility calculated (note how deal with  $c \geq 0$ )
- $\underbrace{T(V(w))}_{TV} = \max_{w'} \underbrace{[u(w - w') + \beta V(w')]}_{Vmat(w, w')}$
- $TV = \max_{w'} Vmat \rightarrow$  get PF here through index of choice of  $w'$
- $TV = V \Rightarrow$  check VFdist =  $TV - V$
- if not, update;  $TV = V \rightarrow$  this is another iteration, do until convergence
- Iterate through problem showing how value function converges, step by step. (e.g. have graph with value function at each iteration).

General Stochastic Dynamic Programming Problem: (similar to above)

- $V(s, \varepsilon) = \max_{s' \in \Gamma(s, \varepsilon)} \tilde{\sigma}(s, s', \varepsilon) + \beta E_{\varepsilon' | \varepsilon} V(s', \varepsilon'), \forall (s, \varepsilon)$
- policy function:  $s' = \phi(s, \varepsilon) \rightarrow$  mapping of  $s$  and  $\varepsilon$  to future state
- $\varepsilon$  follows a first order Markov Process

- transition matrix,  $\Pi$ , which is known,  $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_L\}$ 
  - process is bounded b/c discrete, finite set has max and min
- we could go back through the existence proof with  $\varepsilon$  added and prove unique  $V(s, \varepsilon)$  exists